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Quantum decoherence: a logical perspective

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Abstract The so-called classical limit of quantum mechanics is generally studied in terms of the decoherence of the state operator that characterizes a system. This is not the only possible approach to decoherence. In previous works we have presented the possibility of studying the classical limit in terms of the decoherence of relevant observables of the system. On the basis of this approach, in this paper we introduce the classical limit from a logical perspective, by studying the way in which the logical structure of quantum properties corresponding to relevant observables acquires Boolean characteristics.

Keywords Logic · Decoherence · Lattice · Boolean

1 Introduction

There are different perspectives to address the problem of the classical limit of quantum mechanics. The orthodox treatment introduces the phenomenon of decoherence as the key to solve this problem [1]. The mainstream approach to decoherence is the so-called “environment induced decoherence”, developed by Zurek and his collaborators (see, e.g., [2] [3] [4]). In the context of this approach, the goal is to know whether the state becomes diagonal or not [5]. If the state becomes diagonal, then it acquires the structure of a mixed state of classical mechanics; this feature leads to the usual interpretation of the decohered state from a classical viewpoint.

In our group, we have developed a general theoretical framework for decoherence based on the study of the evolution of the expectation values of

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certain relevant observables of the system [6]. According to this framework, decoherence is a phenomenon relative to the relevant observables selected in each particular case [7].

This new approach and the orthodox treatment of decoherence are equivalent from a mathematical point of view (see [8]). Nevertheless, there are good reasons to think that the treatment of decoherence by means of the behavior of the observables of the system instead of that of its states may have conceptual advantages. For example, it allows us to study the decoherence in closed systems like the universe [9], it does not have the problem of distinguishing between system and environment [7] and this allows us to study open systems without using the reduced state [10].

The purpose of this work is to argue that the main advantage of the study of decoherence in terms of the Heisenberg representation is that this approach allows us to analyze the logical aspects of the classical limit. On the one hand, we know that the lattices of classical properties are distributive or Boolean [11]: when operators are associated with those properties, they commute with each other. On the other hand, it is well known that the lattices of quantum properties are non-distributive, a formal feature manifested by the existence of non-commuting observables [1] [12]. In spite of this difference, there are certain quantum systems which, under certain particular conditions, evolve in a special way: although initially the commutator between two operators is not zero, due to the evolution it tends to become zero [13]. Therefore, in these systems it should be possible to show that, initially, they can be represented by a non-Boolean lattice, but after a definite time a Boolean lattice emerges: this process, that could be described from the perspective of the expectation values of the system's observables, deserves to be considered as a sort of decoherence that leads to the classical limit. In other words, from this perspective the classical limit can be addressed by studying the dynamical evolution of non-Boolean lattices toward Boolean lattices. In the present work we will study this transition from the viewpoint of the general theoretical framework for decoherence.

2 Classical and Quantum Logic

The logical structure of a theory can be studied from the set of properties the theory is able to describe, more specifically, by analyzing under what circumstances an isomorphism between the set of properties and the sentences of language that predicates those properties can be established. If the isomorphism can be consistently established, then the sentences of the language correspond to the properties, and the logical operations on language sentences correspond to certain algebraic operations on the corresponding properties.

The mathematical structure of the properties is not exactly the same as the logical structure of the theory, but by studying the first it is possible to determine the second. The structure of the sentences (propositions) of language can be read and analyzed from the mathematical structure of properties.

When we speak about the properties of the theory, at least in the case of a physical theory, we have to consider the ‘*value properties*’ associated with the physical quantities that the theory describes. So, if an observable \hat{O} of a physical system, with values o_i , acquires the value o_2 when the system is in certain conditions (in some state, say φ), then a value property is represented by the pair definite by $p_2 = (\hat{O} : o_2)$, and the corresponding sentence could be expressed as $L_2 = \text{‘when the system is in state } \varphi, \text{ the magnitude } \hat{O} \text{ has the value } o_2 \text{’}$.

The simplest mathematical structure that can be studied is established with a *partial order relation* between properties. A partial order, \preceq , is an order relation satisfying reflexivity, transitivity and antisymmetry [12]. The order relation between the properties is closely related with the logical entailment between sentences in language. However, not all ordering relations at the level of the properties can be linked to a well-defined entailment. Entailment faces the problem of defining a truth function on of properties (essentially, a function that assigns values 0 and 1 to the properties) that can give rise to a logical structure of sentences. This task can be non trivial in the quantum case [14].

However, even without a well-defined truth function, it is possible to establish a *probability function* on of properties. A probability function \mathcal{P} is a function evaluated on the set of the properties P , which assigns a value between zero and one: $\mathcal{P} : P \rightarrow [0, 1]$. In this case, the link between sentences and properties is established in probabilistic terms. Then, a property represented by $p_2 = (\hat{O} : o_2)$ may correspond to a sentence $L_2 = \text{‘when the system is in the state } \varphi, \text{ the magnitude } O \text{ has the value } o_2 \text{ with probability } \mathcal{P}(p_2) = 0.2$

Endowed with a order relation between properties it is possible to define the operations *meet* \wedge , *join* \vee , and *complement* \perp . These operations respectively correspond to the usual logical connectives between the sentences of the language: conjunction, disjunction and negation [1] [15]. The meet between two properties is defined as the *greatest lower bound* (*GLB*) between them, and the join as the *least upper bound* (*LUB*) [12]. In turn, the complement p^\perp of a property p satisfies $p \wedge p^\perp = 0$ and $p \vee p^\perp = 1$.

When the meet and the join exist for all pairs of the properties, then this defines a *lattice* of properties $R = (P, \preceq)$, where P is the set of all properties, and \preceq is an order relation [12] [15].

If \perp further satisfies

$$(p^\perp)^\perp = p$$

$$p \preceq q \Rightarrow q^\perp \preceq p^\perp$$

then the lattice is said *orthocomplemented*.

The lattice of properties, with the operations meet, join, and complementation representing logical connectives between the corresponding sentences of the language, determines an algebraic structure of properties. This structure characterizes the logical aspects of the theory and allows us to study them.

In the classical case, the set of properties corresponding to the sentences of the language is determined by all the possible subsets of the phase space of

the system, and the partial order relation is given by the inclusion between sets. This leads to a representation of the logical conjunction, disjunction, and negation in the classical discourse by means of the typical operations of intersection, union and complement between sets [1]. The resulting structure determines a Boolean algebra [19]; it is usually said that classical lattices are Boolean lattices [20].

The quantum case is very different. The set of quantum properties is determined by the closed subspaces of the Hilbert space of the system under study [11]. This fact introduces crucial differences in the definition of the operations representing the logical connectives, and has peculiar consequences in the structure of the quantum discourse.

The partial order relation between properties is given by the inclusion of subspaces of Hilbert space. The meet operation is still the intersection, but now between subspaces. The differences are introduced in the join and complementation operations. The join between two properties of a quantum lattice is defined by the closure of the subspace spanned by the linear combinations of the elements of the subspaces representing such properties. That is to say, it is the space spanned by the subspaces of each property [15]. Finally, the complementation of a property is given by the orthogonal complement of the subspace representing that property.

As we have already pointed out, although it is not always possible to establish a well-defined truth function on the lattice of properties, it is nevertheless possible to define a probability function on it, although with certain limitations. Not all probability functions satisfy the axioms of Kolmogorov. The differences among them depend on the Boolean structure (or not) of the lattice [14]. It can be proved that only on Boolean lattices it is possible to introduce a probability function well defined in the Kolmogorovian sense [12] [15] [16] [17] [18]. In more general lattices, as quantum lattices, the probability function is well defined (in the Kolmogorovian sense) only when it is applied on Boolean sublattices.

A simple form of encoding the logical differences between quantum and classical lattices consists in analyzing the validity of the so-called ‘distributive equalities’ [12]. The distributive equalities express the distributivity of the operation meet with respect to the operation join, and vice versa. However, these equalities are not always valid. In general, only distributive inequalities hold. Given the properties a , b and c , the following inequalities are always valid

$$\begin{aligned} a \wedge (b \vee c) &\succeq (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &\preceq (a \vee b) \wedge (a \vee c) \end{aligned}$$

Only in a Boolean lattice the equalities hold.

Another important aspect associated with distributive inequalities is that they capture the notion of *compatibility* as understood in quantum mechanics [12]. It can be proved that, if the properties a and b are such that

$$\begin{aligned} a &= (a \wedge b) \vee (a \wedge b^\perp) \\ b &= (b \wedge a) \vee (b \wedge a^\perp) \end{aligned}$$

then the projectors associated with the subspaces representing these properties commute. Otherwise, the projectors do not commute and their value properties are incompatible. Of course, if those projectors are involved in the spectral decomposition of the observables A and B , then A and B are also incompatible observables.

We arrive thus to an important conclusion. Only when all the properties to be described are associated with compatible observables, there is a Boolean structure corresponding to a classical description, and in this case the distributive equalities hold and the probabilities are well defined in Kolmogorovian sense. Otherwise, there are incompatible observables, the lattice structure is not Boolean, and the probabilities, in general, are not well defined in the Kolmogorovian sense.

3 Incompatibility of observables in time

As it is well known, quantum mechanics admits at least two representations. The Schrödinger representation studies the evolution of the state $\hat{\rho}(t)$, and the Heisenberg representation studies the evolution of the observables $\hat{O}(t)$ [21]. The traditional approach to decoherence emphasizes the evolution of the state in the Schrödinger representation: it studies the diagonalization of the state in the preferred basis [22] [4]. Such diagonalization removes interference, which is one of the phenomena specific of quantum mechanics. However, this approach does not make explicit the disappearance of another peculiar feature of quantum mechanics, that is, contextuality. Contextuality is linked to the non-commutativity of observables, because two non-commuting observables belong to different contexts. The Heisenberg uncertainty principle is other manifestation of non-commutativity, and expresses the fact that it is not possible to simultaneously measure the value of two non-commuting observables. This principle establishes a fundamental difference with classical mechanics, where all the observables commute with each other. Therefore, any attempt to construct a classical limit should include a mechanism capable of explaining the transition from non-commutativity to commutativity.

In the Schrödinger picture, if a pair of observables do not commute in the initial time,

$$[\hat{O}_1, \hat{O}_2] \neq 0$$

then they do not commute ever, since observables do not evolve. For this reason, the natural picture to study the transition from non-commutativity to commutativity is the Heisenberg picture. Some authors, like Kiefer and Polarski, described decoherence in the Heisenberg representation [13], [23]. In the present paper, our purpose is to continue this line of work by studying the time evolution of the logical properties of quantum systems. Our aim is to find a process in which two observables do not commute at the initial time, but they do commute later:

$$[\hat{O}_1(0), \hat{O}_2(0)] \neq 0 \longrightarrow [\hat{O}_1(t), \hat{O}_2(t)] \cong 0$$

For this purpose, we will use the approach to decoherence called "Self-Induced Decoherence" (SID), developed in our group [6] [8] [9] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37]. This approach will allow us to easily show the process of interest. Nevertheless, the same result can be obtained by the orthodox 'Environment Induced Decoherence' approach.

3.1 Self-Induced decoherence in the Heisenberg picture

Although at present EID is still considered the "orthodoxy" in the subject [1], other approaches have been proposed to face its problems, in particular, the closed-systems problem. One of them is SID, according to which a closed quantum system with continuous spectrum may decohere by destructive interference and reach a final state where the classical limit can be rigorously obtained [9] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37].

Self-Induced decoherence (SID) is a formalism that finds its roots in an algebraic formalism which was initiated in the Brussels school, lead by Prigogine [38]. In this paper we will use the notation according to which the observables are thought as vectors, and we write them as $\hat{O} = |O\rangle$. SID considers a closed quantum system governed by a Hamiltonian with continuous spectrum. Then we can write a generic observable as $\hat{O} = \int \int O(\omega\omega') |\omega\omega'\rangle d\omega d\omega'$ where $O(\omega\omega')$ is a generic distribution, and $|\omega\omega'\rangle$ are generalized eigenvectors of space observable, that is, $\{|\omega\omega'\rangle\}$ is the basis of space. This notation is necessary for technical reasons we will not discuss in this article but can be found in [38].

According to the work of our group, the different approaches to decoherence can be described from a General Theoretical Framework for Decoherence consisting of 3 steps [6] [8] [36] [37]. The most important step is to choose a subset of the observables of interest.

EID adopts the open system prespective, that is, the relevant observables have the form $\hat{O}_R = \hat{O}_S \otimes \hat{I}_E$, where \hat{I}_E is the identity in the space of the environment and \hat{O}_S is any observable of the proper system. On the other hand SID selects the van Hove observables as relevant observables, it is a good choice because the restriction on the observables does not diminish the generality of this approach, because the observables not belonging to the van Hove space are not experimentally accessible. Then, if we compute the evolution of the mean values of the van Hove observables, we find that the interference terms disappear. Then it is possible to understand this process as a kind of decoherence [6]. You can find an exhaustive comparison between SID and EID in the paper [37]. Here we present a version of Self-Induced decoherence in the Heisenberg picture.

In this case, we will select special observables that are appropriate to our study, i.e., the time evolution of commutators. In SID, we consider a quantum system with Hamiltonian H with continuous spectrum: $H|\omega\rangle = \omega|\omega\rangle$, $\omega \in [0, \infty)$. Thus, the three steps are:

First step: Selection of observables. At $t = 0$, a generic observable can be written as

$$\hat{O}(0) = \int_0^\infty \int_0^\infty \tilde{O}(\omega, \omega') |\omega, \omega'\rangle d\omega d\omega' \quad (1)$$

where $\tilde{O}(\omega, \omega')$ is any core or distribution. We will consider only the *van Hove observables* [39] [40], which have a core $\tilde{O}(\omega, \omega')$ of the form:

$$\tilde{O}_{vH}(\omega, \omega') = O(\omega)\delta(\omega - \omega') + O(\omega, \omega') \quad (2)$$

where $O(\omega, \omega')$ is a regular function. Therefore, the van Hove observables have the form:

$$\hat{O}_{vH}(0) = \int_0^\infty O(\omega) |\omega\rangle d\omega + \int_0^\infty \int_0^\infty O(\omega, \omega') |\omega, \omega'\rangle d\omega d\omega' \quad (3)$$

These observables belong to *van Hove space* \mathcal{O}_{vH} , whose basis is $\{|\omega\rangle, |\omega, \omega'\rangle\}$. This restriction on the observables does not diminish the generality of SID, because the observables not belonging to the van Hove space are not accessible to experiments [41]. The states $\hat{\rho}$, which do not evolve in the Heisenberg picture, are represented by linear functionals on \mathcal{O}_{vH} , that is, they belong to the dual space \mathcal{O}'_{vH} and can be written as:

$$\hat{\rho} = \int_0^\infty \rho(\omega) (\omega| d\omega + \int_0^\infty \int_0^\infty \rho(\omega, \omega') (\omega, \omega'| d\omega d\omega' \quad (4)$$

where $\{(\omega|, (\omega, \omega'|)\}$ is the co-basis of $\{|\omega\rangle, |\omega, \omega'\rangle\}$, that is, the basis of \mathcal{O}'_{vH} . States must satisfy the usual requirements, i.e., $\rho(\omega)$ is real and positive, and $\int_0^\infty \rho(\omega) d\omega = 1$. It is also required that $\rho(\omega, \omega')$ be a regular function. Under these conditions, the states belong to a convex set $S \subset \mathcal{O}'_{vH}$.

According to the Heisenberg picture, the evolution of the observables is given by

$$\hat{O}(t) = e^{i\hat{H}t} \hat{O}(0) e^{-i\hat{H}t}$$

Then, expression (3) becomes

$$\hat{O}_{vH}(t) = \int_0^\infty O(\omega) |\omega\rangle d\omega + \int_0^\infty \int_0^\infty O(\omega, \omega') e^{i(\omega - \omega')t} |\omega, \omega'\rangle d\omega d\omega' \quad (5)$$

Now we select a subset of the van Hove space, defined by the commutators. The commutator of any two observables $\hat{O}_1(t) \in \mathcal{O}_{vH}$ and $\hat{O}_2(t) \in \mathcal{O}_{vH}$ is

$$[\hat{O}_1(t), \hat{O}_2(t)] = \hat{O}_1(t)\hat{O}_2(t) - \hat{O}_2(t)\hat{O}_1(t)$$

By operating with patience, we obtain:

$$\hat{C}(t) = [\hat{O}_1(t), \hat{O}_2(t)] = \int_0^\infty \int_0^\infty C(\omega, \omega') e^{i(\omega - \omega')t} (\omega, \omega'| d\omega d\omega'$$

where

$$C(\omega, \omega') = \int_0^\infty (O_1(\omega, \tilde{\omega}') O_2(\tilde{\omega}', \omega') - O_2(\omega, \tilde{\omega}') O_1(\tilde{\omega}', \omega')) d\tilde{\omega}'$$

It is important to notice that $\hat{C}(t) \notin \mathcal{O}_{VH}$ because it is not an Hermitian operator. However, the observable $\hat{D}(t) = i\hat{C}(t) \in \mathcal{O}_{VH}$ is a legitimate quantum observable, to which we may have empirical access.

The observable $\hat{D}(t)$ allows us to measure the degree of incompatibility between the observables $\hat{O}_1(t)$ and $\hat{O}_2(t)$. For example, it can be the observable that measures the contrast between the interference fringes in the double slit experiment. This contrast indicates that the observable that measures by which slit the particle passes is incompatible with the observable that measures where on the screen the particle impacts. Then, the relevant observables considered here are the observables $\hat{D}(t)$.

Second step: The computation of the expectation value. We consider the observable \hat{D} at $t = 0$

$$\hat{D}(0) = i^{-1} [\hat{O}_1(0), \hat{O}_2(0)] = i^{-1} \int_0^\infty \int_0^\infty C(\omega, \omega') (\omega, \omega' | d\omega d\omega'$$

Then we assume that the initial commutator is not 0, i.e. $\hat{O}_1(0)$ and $\hat{O}_2(0)$ do not commute

$$\hat{C}(0) = [\hat{O}_1(0), \hat{O}_2(0)] \neq 0 \longrightarrow \hat{D}(0) \neq 0$$

This means that $C(\omega, \omega')$ is a nonzero function. The expectation value of $\hat{D}(t)$ is

$$\langle \hat{D}(t) \rangle_\rho = \text{Tr}(\rho \hat{D}(t))$$

that is

$$\langle \hat{D}(t) \rangle_\rho = \langle i^{-1} [\hat{O}_1(t), \hat{O}_2(t)] \rangle_\rho = i^{-1} \int_0^\infty \int_0^\infty \rho(\omega, \omega') C(\omega, \omega') e^{i(\omega - \omega')t} d\omega d\omega'$$

Third step: The evolution of the expectation value. We assume that $\rho(\omega, \omega') C(\omega, \omega')$ is a regular function, indeed simply a L_1 function in the variable $\nu = \omega - \omega'$; then, the Riemann-Lebesgue theorem can be applied. Consequently,

$$\lim_{t \rightarrow \infty} \langle \hat{D}(t) \rangle_\rho = 0 \quad (6)$$

This means that, when $t \rightarrow \infty$, the expectation value of the commutator between $\hat{O}_1(0)$ and $\hat{O}_2(0)$ becomes zero. Therefore, the Heisenberg uncertainty relation becomes undetectable from the experimental viewpoint.

In other words, when $t \rightarrow \infty$ we can compute the expectation value of $\hat{D}(t)$ for any ρ as follows. We may think that the observable $\hat{D}(t)$ is a final fixed observable $\hat{D}(*)$ such that

$$\lim_{t \rightarrow \infty} \langle \hat{D}(t) \rangle_\rho = \langle \hat{D}(*) \rangle_\rho \quad (7)$$

where $\hat{D}(*) = 0$. This result can also be expressed as a weak limit:

$$W - \lim_{t \rightarrow \infty} \hat{D}(t) = 0 \quad (8)$$

In this way we arrive closer to the classical limit. Interference is not the only quantum feature that vanishes: from the experimental viewpoint, the initially non-commuting observables, tend to commute after a sufficient time.

4 Classical limit in the logical structure

The central task of this work is to study the classical limit of quantum mechanics from the point of view of the logical structure of the theory. We have already seen that the essential difference between the lattice of the classical properties and the lattice of the quantum properties is that in the first one the distributive equalities hold. Only in a distributive and ortocomplemented lattice we have a Boolean structure of the properties.

Therefore, the study of the classical limit requires the study of under what conditions a quantum structure of properties becomes Boolean. It is clear that this limit must involve a non-unitary evolution, a coarse-grained [8], or some additional element; otherwise, a set of properties whose projectors do not commute, and therefore form a non-classical algebra, will never lose this feature. But we have seen that evolutions of this type are involved in the search of the classical limit as a result of the mechanism of decoherence .

The decoherence studied in the previous sections meets our goal. In fact, on the basis of the evolutions studied here, it is possible to show that the commutator between certain observables $\hat{O}_1(t)$ and $\hat{O}_2(t)$ vanishes, at least in terms of their expectation values. Therefore, if we measure the observable $\hat{D}(t)$ at the beginning of the process, its expectation value is not zero; but if we measure this observable at the end of the process, its expectation value is almost zero. This means that, from the observational point of view, we can assume that \hat{O}_1 and \hat{O}_2 are compatible observables. But, does this mean that now we have recovered distributivity?

We can interpret the evolution of these observables as follows. Let us consider two properties, A corresponding to the value o_1 of the observable \hat{O}_1 , and B corresponding to the value o_2 of the observable \hat{O}_2 . If we think these observable properties as vectors in the Hilbert space, then they enclose an angle. The evolution is such that the angle between the vectors representing the properties “ o_1 ” and “ o_2 ” gets smaller. While the angle is not exactly zero, we have non-distributivity. But in the infinite time limit, the angle between the vectors representing the properties A and B becomes zero, and the corresponding observables turn out to commute with each other. Therefore, distributivity is recovered.

In other words, decoherence can be also viewed as a process that turns incompatible observables into compatible observables and, as a consequence, that turns the quantum logic into the classical Boolean logic.

5 Conclusion

Through the decoherence of the expectation values it is possible to study the classical limit of a quantum system in terms of decohering observables. These are the relevant observables of the system, and from their property values it is possible to construct the logical structure of interest.

This endows decoherence with a semantic content stronger than that involving the mere process by which the interference terms vanish. The evolution of the commutators allows us to understand decoherence as a process by which the logical structure of what can be said about the system acquires classical characteristics, i.e., becomes Boolean. These features have relevant consequences on the calculation of the probabilities of the values of the decohering observables.

Therefore, we can establish the transition between two logics, quantum logic and classical logic, from the observational point of view. We propose to continue this line of work by studying the evolution of the logical properties of the system in time, for example, by analyzing the evolution of the observables, not of their the expectation values. On the other hand, although the usual lattice is constructed from properties, we can try to build a lattice from expectation values. In both cases, we could describe the transition from quantum logic to Boolean logic.

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